# A BRIEF INTRODUCTION TO CATEGORY THEORY

AKASH DHIRAJ

ABSTRACT. We hope to discuss various category theoretical concepts. To begin, we briefly introduce *Categories*, providing familiar examples. Following which, we discuss the various types of *Morphisms* and a few examples of *Universal Properties*. Along the way, we prove an analogue of *Cayley's Theorem* for categories. One point of interest is the *Yoneda Lemma*. We discuss this after a quick introduction to *Functors* and *Natural Transformations*.

It's the arrows that really matter!

*Steve Awodey*

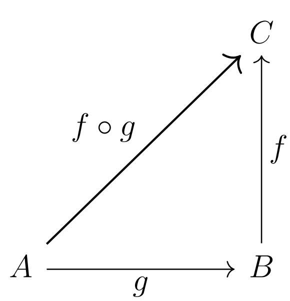
This paper is written for an audience familiar with basic algebra. The unfamiliar reader should consult [DF04].

## 1. What is a Category?

As expected, we begin by defining a *Category*.

**Definition 1.1** (Category)**.** A category $\mathcal{C}$ consists of a collection of *objects* denoted $\mathrm{Obj}(\mathcal{C})$. For all $A, B \in \mathrm{Obj}(\mathcal{C})$, there exists a collection of *morphisms*[1] from $A$ to $B$ (denoted $\mathrm{Hom}_{\mathcal{C}}(A, B)$) with special properties:

- Certain morphisms can be composed to produce new morphisms. Specifically, for objects $A, B, C$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, there exists an element of $\mathrm{Hom}_{\mathcal{C}}(A, C)$ denoted $gf$ or $g \circ f$ (read as $f$ composed with $g$). I.e. there exists a function
$$\phi : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C).$$
This composition can be represented using the following commutative diagram:




- Morphism composition is associative. That is, for objects $A, B, C, D$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$,
$$(hg)f = h(gf).$$

*Date*: July 31, 2020.

[1]Morphisms may also be called arrows. We use these two words interchangeably

- There exists identity elements under composition. I.e. for objects $A$ and $B$, there exists $1_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ and $1_B \in \mathrm{Hom}_{\mathcal{C}}(B,B)$ such that $\forall f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, $f \circ 1_A = f$ and $1_B \circ f = f$.

**Remark.** We use $f : A \to B$ interchangeably with $f \in \mathrm{Hom}(A,B)$.

**Remark.** We use "collection" to avoid size issues. Categories are, in general, considered 'bigger' than sets.

**Proposition 1.2.** *For any object $A$ of a category $\mathcal{C}$, the identity morphism of $\mathrm{Hom}_{\mathcal{C}}(A,A)$ is unique.*

*Proof.* While apparent, we'll prove this for completeness. Consider identity morphisms $1_A, 1'_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$. Then, $1_A = 1_A 1'_A = 1'_A$. ∎

Categories are **everywhere**.

**Exercise.** Before reading some of the examples below, you should try to come up with some categories of your own. What's the most interesting example of a category you can come up with?

Now for some familiar (and not-so familiar) examples of categories.

**Example 1.3.** The prototypical example of a category is $\mathcal{C} = \mathbf{Set}$. When imagining categories, **Set** is a good picture to have in mind. Let $\mathrm{Obj}(\mathcal{C})$ be the collection of all sets, and, for sets $A$ and $B$, let $\mathrm{Hom}_{\mathcal{C}}(A,B)$ be functions $f : A \to B$. Verifying that all our axioms hold is fairly easy. Naturally, we define the composition of morphisms as function composition. Since we have our identity maps (the map from a set to itself that fixes every element) and function composition is associative, we conclude that **Set** is indeed a category.

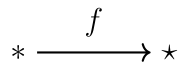
**Example 1.4.** In a similar vein, we have the categories

(1) **Grp**, where our objects are groups and our morphisms are group homomorphisms.
(2) **CRing**, where our objects are commutative rings and our morphisms are ring homomorphisms.
(3) **Top**, where our objects are topological spaces and our morphisms are continuous functions.
(4) $\mathbf{Vect}_k$, where are objects are vector spaces over a field $k$ and our morphisms are linear transformations.
(5) **Poset**, where our objects are partially ordered sets and our morphisms are monotone functions.

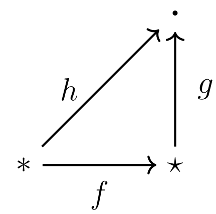
Now for some less concrete[2] examples.

**Example 1.5.** Consider an arbitrary poset $S$. We'll build a category $\mathcal{C}$ out of it. Let $\mathrm{Obj}(\mathcal{C}) = S$. For $p, q \in S$, if $p \leq q$, then a single arrow exists between $p$ and $q$. Otherwise, $\mathrm{Hom}_{\mathcal{C}}(p,q) = \varnothing$. Since we only have one morphism between any two objects, there is only one way composition can work. For another object $r \in S$, suppose $f \in \mathrm{Hom}_{\mathcal{C}}(p,q)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(q,r)$. Since $p \leq q$ and $q \leq r$ implies $p \leq r$, we know there exists unique $h \in \mathrm{Hom}_{\mathcal{C}}(p,r)$. So, set $gf := h$. We leave verifying that we have an identity and that morphism composition is associative for the reader (notice that we haven't actually used antisymmetry, so this works, more generally, for a set with a transitive and reflexive relation).

[2]Curiously, a *concrete* category has a precise meaning. Think of it as meaning the categories where the underlying objects are sets with additional structure. For a precise definition see this




(A) Category **2**

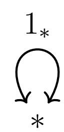



(B) Category **3**

FIGURE 1. Categories **2** and **3**

The moral of this story is that, unlike many of the examples we've seen before, morphisms don't actually have to be some form of maps. And, our objects don't have to have some underlying set structure. Still, we encourage retaining this mental image as it rarely causes any problems.

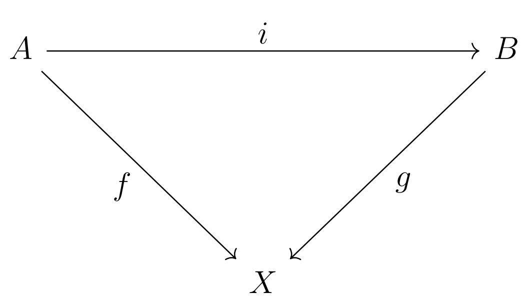
**Example 1.6** (Finite Categories)**.** A category is finite when its underlying set of morphisms and objects are both finite sets. Consider category **1**, the one object category with only the identity morphism. See the following diagram:




Similarly, consider the categories **2** and **3**, represented by the commutative diagrams in Figure 1. (we omit the identity morphisms.)

**Example 1.7** (Opposite Category)**.** For a category $\mathcal{C}$, consider the *opposite category*, denoted $\mathcal{C}^{\text{op}}$. $\text{Obj}(\mathcal{C}^{\text{op}}) := \text{Obj}(\mathcal{C})$, but we reverse the direction of our arrows. So, for objects $A, B$, $\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) := \text{Hom}_{\mathcal{C}}(B, A)$. So, how do we define composition? To make sure we don't confuse when we're working in $\mathcal{C}$ or $\mathcal{C}^{\text{op}}$, lets denote $\circ$ and $*$ for composition in $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ respectively. For another object $C \in \text{Obj}(\mathcal{C}^{\text{op}})$, consider $f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, C)$. Naturally, $f * g := g \circ f$. We note that the existence of identity morphisms and the associativity of morphism composition follows since $\mathcal{C}$ is a category.

**Example 1.8** (Slice Category)**.** Consider a category $\mathcal{C}$ and an object $X \in \text{Obj}(\mathcal{C})$. We construct the *slice category* $\mathcal{C}/X$ such that our objects are morphisms $f \in \text{Hom}_{\mathcal{C}}(Y, X)$ for all $Y \in \text{Obj}(\mathcal{C})$. Consider objects $f : A \to X$ and $g : B \to X$ of $\mathcal{C}/X$. The elements of $\text{Hom}_{\mathcal{C}/X}(f, g)$ are morphisms $i$ of $\mathcal{C}$ that make the diagram




commute. Now, for morphisms $i \in \text{Hom}_{\mathcal{C}/X}(f, g)$ and $j \in \text{Hom}_{\mathcal{C}/X}(g, h)$, we'll set $i * j := i \circ j$, where $*$ and $\circ$ indicate morphism composition in $\mathcal{C}/X$ and $\mathcal{C}$ respectively. The associativity of $*$ follows from that of $\circ$, and the existence of an identity in $\mathrm{Hom}_{\mathcal{C}/X}(f, f)$ follows from the existence of an identity in $\mathrm{Hom}_{\mathcal{C}}(A, A)$.

**Example 1.9** (Product Category)**.** Consider two categories $\mathcal{A}$ and $\mathcal{B}$. We construct the *product category* $\mathcal{A} \times \mathcal{B}$, where

$$\mathrm{Obj}(\mathcal{A} \times \mathcal{B}) \coloneqq \mathrm{Obj}(\mathcal{A}) \times \mathrm{Obj}(\mathcal{B}) \coloneqq \{(A, B) \mid A \in \mathrm{Obj}(\mathcal{A}), B \in \mathrm{Obj}(\mathcal{B})\}.$$

Similarly, for objects $(A, B)$ and $(C, D)$, the morphisms $f : (A, B) \to (C, D)$ are arrows $f = (g, h)$, where $A \overset{g}{\mapsto} C$ and $B \overset{h}{\mapsto} D$.

**Exercise.** How do you think composition should work? What should our identities look like?

## 2. Morphisms

From algebra, you might recall that isomorphisms are just morphism that are bijective. So, we might be tempted to define isomorphisms in categories similarly. But, bijections don't make sense for general categories as our morphisms aren't actually maps (Example 1.5 and 2.3). We bypass this by use of an alternative definition for isomorphism that captures the properties we desire: namely, the existence of an inverse.

**Definition 2.1** (Isomorphism)**.** For objects $A, B$ of a category $\mathcal{C}$, the morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is said to be an *isomorphism* if it has a two sided inverse, i.e. there exists $f^{-1} \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that

$$f \circ f^{-1} = 1_B, \tag{1}$$

$$f^{-1} \circ f = 1_A. \tag{2}$$

We call $f^{-1}$ the inverse of $f$. And, when an isomorphism exists between objects $A$ and $B$, we say $A \cong B$.

Hopefully, you can see that in **Set**, **Grp**, and **CRing**, our isomorphisms are indeed bijective morphisms.

**Proposition 2.2.** *For objects $A, B$ of a category $\mathcal{C}$ and morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $f^{-1}$ is unique if it exists.*

*Proof.* Once again, this result is apparent, but we'll prove it for completeness. Let $g, h \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ be inverses of $f$. Then, using associativity, $g = gfh = h$. ■

With our new notion of 'isomorphism', let's consider what a category with a single object, for which all morphisms are isomorphisms, looks like. Perhaps this seems like an odd category to consider, but it actually has a structure that we're all quite familiar with.

**Example 2.3.** The data of a category with a single object, where every morphism is an isomorphism, is a group. That is, for a category $\mathcal{X}$, where $\mathrm{Obj}(\mathcal{X}) = \{*\}$ and all morphisms of $\mathrm{Hom}_{\mathcal{X}}(*, *)$ are isomorphism, $\mathrm{Hom}_{\mathcal{X}}(*, *)$ forms a group under morphism composition. Similarly, for a group $G$, we can construct a category $\mathcal{X}$ with one object $*$ such that $\mathrm{Hom}_{\mathcal{X}}(*, *) \coloneqq G$. Morphism composition will work as expected. For morphisms $g, h \in G$, $g \circ h \coloneqq gh$.

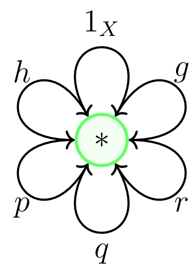



FIGURE 2. One Object Categories as Groups

Your intuition may be raising a few alarms since we've talked about isomorphisms without discussion of any analogue of injectivity and surjectivity yet. Surely there is a good analogue of this in the world of category theory. In classic category theoretical fashion, we presume no knowledge of the objects/morphisms we work with so our usual set theoretical definitions don't translate well again.

**Definition 2.4** (Monomorphism)**.** For a category $\mathcal{C}$, objects $A, X, Y \in \text{Obj}(\mathcal{C})$ and arrows $f, g, h$, an arrow $f : X \to Y$ is *Monic* (or a *Monomorphism*) if, for any commutative diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{\quad f \quad} Y,$$

we must have that $g = h$.

**Definition 2.5** (Epimorphism)**.** For a category $\mathcal{C}$, objects $A, X, Y \in \text{Obj}(\mathcal{C})$ and arrows $f, g, h$, an arrow $f : X \to Y$ is *Epic* (or an *Epimorphism*) if, for any commutative diagram

$$X \xrightarrow{\quad f \quad} Y \underset{h}{\overset{g}{\rightrightarrows}} A,$$

we must have that $g = h$.

Monomorphisms and epimorphisms are our category theoretical versions of injections and surjections respectively. And, as expected,

**Proposition 2.6.** *In a category $\mathcal{C}$ with objects $A, B$, an arrow $f : X \to Y$ is both epic and monic if it's an isomorphism*

*Proof.* Since $fg = fh$, we have that $g = f^{-1}fg = f^{-1}fh = h$. Similarly, we note that $gf = hf \implies g = gff^{-1} = hff^{-1} = h$. ■

So, when does the converse hold? It holds when we're in **Set**, but, in general, it fails quite miserably. For example, in **Top**, consider the morphism $f : X \to Y$, where $X = [0, 1]$ has the standard topology on $\mathbb{R}$, $Y = [0, 1]$ has the discrete topology, and $f(x) = x$.

## 3. UNIVERSAL PROPERTIES

**Definition 3.1** (Universal Property)**.** Let $\mathcal{C}$ be a category. Let $X \in \text{Obj}(\mathcal{C})$ be an object and $\{f_i\}_{i \in I}$ a collection of morphisms. We say $X$ has a *universal property* if, for every other $Y \in \text{Obj}(\mathcal{C})$ and morphisms $\{g_j\}_{j \in J}$ from/to the same objects, we can find $h : X \to Y$ such that the $g_i$'s can be obtained with compositions of the $f_i$'s with $h$.

Intuitively, we think of universal properties as a way to construct an object with morphisms that is unique up to isomorphism. This is particularly nice because it means we can avoid discussion of the precise structure of the object. Saying what the object "does" ends up being enough.

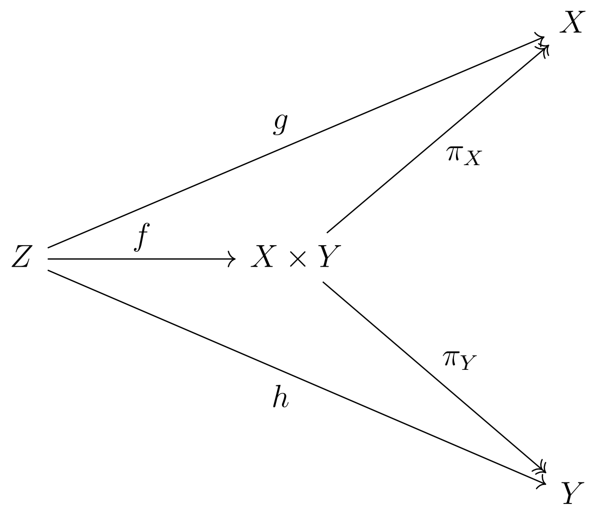
Now for some examples!

3.1. **Binary Product.** Recall that the *Cartesian Product* of sets $A, B$ is

$$A \times B \coloneqq \{(a, b) \mid a \in A, b \in B\}.$$

Can we extend this notion of *product* beyond **Set** and to more general categories? Yes!

**Definition 3.2** (Binary Product)**.** In a category $\mathcal{C}$, consider objects $X$, $Y$. The product of $X$,$Y$ consists of an object denoted $X \times Y$ and a pair of *projection morphisms* $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ such that for any object $Z \in \mathrm{Obj}(\mathcal{C})$ and arrows $g : Z \to X$, $h : Z \to Y$, there *exists unique*[3] $f : Z \to X \times Y$ for which the diagram




commutes.

**Remark.** We note that our binary product is dependent on not only our choice of $X$ and $Y$ but also our choice of $\pi_X$ and $\pi_Y$.

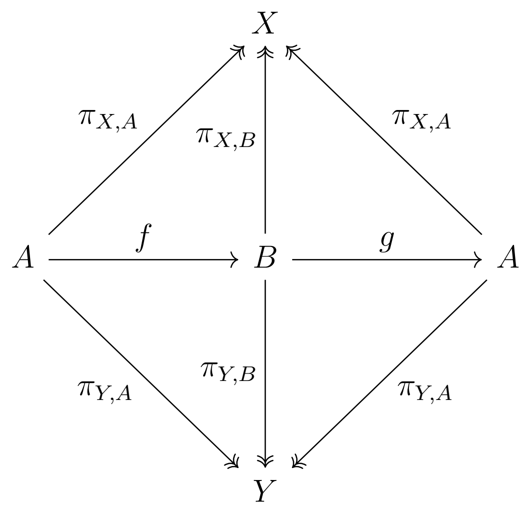
**Remark.** We note that a binary product need not always exist. As an obvious example, consider the category with precisely two objects $X, Y$ and no non-identity morphisms. Since we lack non-identity morphisms, there are no good candidates for $X \times Y$ since we can't satisfy the existence part of Definition 3.2.

Two natural question to ponder come up rather quickly. Firstly, how is this related to our usual cartesian product? And, secondly, does this definition actually uniquely define the product of objects? That is, can we find two different objects $A, B$ that can both be consider $X \times Y$? Suppose $X$ and $Y$ were sets and we chose $\pi_X$ and $\pi_Y$ such that $\pi_X(a, b) = a$ and $\pi_Y(a, b) = b$. Then, we obtain our regular cartesian product. For the second question, it turns out we can find different $A$ and $B$, but $A \cong B$.

**Proposition 3.3.** *Given two binary products $A$ and $B$ of $X, Y$ (not necessarily with the same projection morphisms), then $A \cong B$.*

[3]There's two important parts of this definition: the existence and uniqueness of $f$

*Proof.* Consider the following diagram:




The existence of $f$ and $g$ follows from Definition 3.2. Then, $fg$ makes the outer square commute. Since both $fg$ and $1_A$ do this, the uniqueness part of Definition 3.2 guarantees $fg = 1_A$. Applying a similar argument, we can show that $gf = 1_B$. ∎

This proposition is particularly nice because it means we don't actually have to provide an explicit construction of the product of objects. Since we've shown that our definition restricts our choices of $X \times Y$ to isomorphic objects, all our discussion can be up to isomorphism.

Now for some special objects.

### 3.2. Initial & Terminal Objects.

**Definition 3.4** (Initial Objects)**.** In a Category $\mathcal{C}$, an objects $X \in \text{Obj}(\mathcal{C})$ is an *initial object* if, for every $Y \in \text{Obj}(\mathcal{C})$, there exists precisely one arrow in $\text{Hom}_{\mathcal{C}}(X, Y)$.

**Definition 3.5** (Terminal Objects)**.** In a Category $\mathcal{C}$, an objects $X \in \text{Obj}(\mathcal{C})$ is a *terminal object* if, for every $Y \in \text{Obj}(\mathcal{C})$, there exists precisely one arrow in $\text{Hom}_{\mathcal{C}}(Y, X)$.

**Example 3.6.** Some initial objects:

(1) In **Set**, it's $\varnothing$.
(2) In **Grp**, it's $\{e\}$, the trivial group.
(3) In the category of Example 1.5, it's the minimal element if it exists.

**Example 3.7.** Some terminal objects:

(1) In **Set**, it's $\{1\}$.
(2) In **Grp**, it's $\{e\}$, the trivial group.
(3) In the category of Example 1.5, it's the maximal element if it exists.

Similar to products, initial and terminal objects are unique up to isomorphism.

**Proposition 3.8.** *If $X, Y \in \text{Obj}(\mathcal{C})$ are initial objects of a category $\mathcal{C}$, $X \cong Y$.*

*Proof.* As $X, Y$ are initial, there exists unique $f : X \to Y$ and $g : Y \to X$. Since $f \circ g \in \text{Hom}_{\mathcal{C}}(Y, Y)$ and $g \circ f \in \text{Hom}_{\mathcal{C}}(X, X)$, $f \circ g = 1_Y$ and $g \circ f = 1_X$. ∎

**Proposition 3.9.** *If $X, Y \in \text{Obj}(\mathcal{C})$ are initial objects of a category $\mathcal{C}$, $X \cong Y$.*

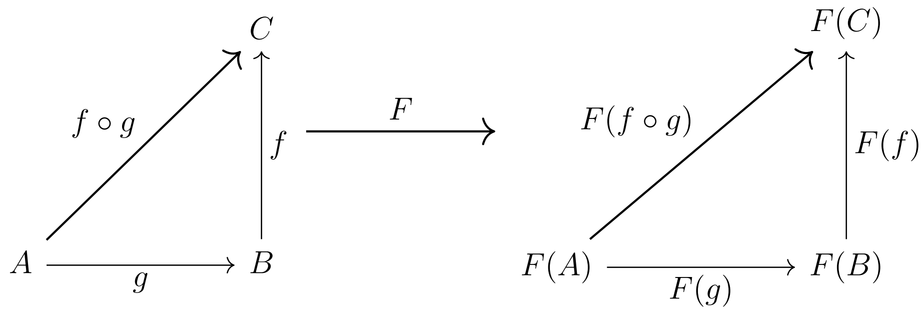



FIGURE 3. $F : \mathcal{A} \to \mathcal{B}$ is a functor and $A, B, C \in \mathrm{Obj}(\mathcal{A})$

*Proof.* Apply a similar argument. Filling out the details is left to the reader. Alternatively, a nice trick we could use is to note that the terminal objects of $\mathcal{C}$ are simply the initial objects of $\mathcal{C}^{\mathrm{op}}$. Then, by Proposition 3.8, we use that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ to conclude the desired result. ∎

## 4. Functors & Natural Transformations

A foundational philosophy of category theory is the importance of the arrows between our objects over the objects themselves. So, the natural question to ask is what the morphisms between categories are.

### 4.1. **Covariant Functors.**

**Definition 4.1** (Covariant Functor)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *covariant functor* $F : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ is a map between the objects and arrows of $\mathcal{A}$ and $\mathcal{B}$ satisfying the following properties: for objects $X, Y$ and arrows $f, g$ of $\mathcal{A}$,

(1) $F(f : X \to Y) = F(f) : F(X) \to F(Y)$,
(2) $F(1_X) = 1_{F(X)}$,
(3) and $F(g \circ f) = F(g) \circ F(f)$.

These properties are called the *naturality requirements*. See Figure 3 for a diagrammatic representation.

Let's look at some examples!

**Example 4.2.** Consider the *endofunctor*[4] $F : \mathbf{Set} \to \mathbf{Set}$, where $F$ sends all sets to their power sets. And, for $f : A \to B$ for sets $A, B$, $F(f) = f_*$, where $f_* : F(A) \to F(B)$ and $X \mapsto_{f_*} f(X)$.

**Example 4.3** (Forgetful Functor)**.** Consider a category $C$, whose objects are some form of sets: **Grp**, **CRing**, or $\mathbf{Vect}_k$ for example. A *forgetful functor* is a general term for a functor that forgets some of the structure of an object. We can construct a forgetful functor from $C$ to **Set** by sending each object to its underlying set and every morphism to its underling map.

**Example 4.4.** Similar to the previous example, we can construct an intermediate version of forgetful functor. For example, consider the functor that sends the objects of **CRing** to **Ab**, the category of abelian groups.

[4]An endofunctor is a functor from a category to itself

**Remark.** It's worth noting that forgetful functors have no technical definition. It's simply a label we use for functors that intuitively appear to forget structure.

**Example 4.5.** Remember Example 2.3? Let $G$ be a group and $\mathcal{X}$ the corresponding one object category, with object $*$.

(1) Let $F : \mathcal{X} \to \mathbf{Set}$ be a functor. Then, the data of $F$ corresponds to a group action on $S \coloneqq F(*)$. Recall that a group action of $G$ on $S$ is a map $\phi : G \times S \to S$ that satisfies
   (a) $\forall s \in S$, $\phi(e, s) = s$,
   (b) and $\forall g, h \in G$, $\forall s \in S$, $\phi(g, \phi(h, s)) = \phi(gh, s)$.
   Now, we'll construct $\phi$ out of $F$ such that $\phi(g, s) = (F(g))(s)$. Do (a) and (b) follow from the naturality requirements? For the other direction, we'll construct $F$ out of some group action $\phi$ on $S$. Set $F(*) = S$ and set $F(g)$ to the automorphism of $S$ induced by the action of $g$. Does this satisfy the naturality requirements?

(2) Let $H$ be a group and $\mathcal{Y}$ the corresponding one object category, with object $\star$. Then, the functors from $\mathcal{X}$ to $\mathcal{Y}$ correspond to morphisms from $G$ to $H$. Let's consider a functor $F : \mathcal{X} \to \mathcal{Y}$. $F$ provides a map between the morphisms of $X$ and $Y$, which we may use as a homomorphism from $G$ to $H$. Naturality requirements (2) and (3) guarantee our map between morphisms is indeed a homomorphism. To build a functor from $\mathcal{X}$ to $\mathcal{Y}$ out of a homomorphism from $G$ to $H$, it suffices to use the homomorphism as our map between morphisms. Then, we'll send $*$ to $\star$. Do we satisfy the naturality requirements?

**Example 4.6** (Covariant Yoneda Functor)**.** Consider a category $\mathcal{C}$, where, for all $A, B \in \text{Obj}(\mathcal{C})$, $\text{Hom}_{\mathcal{C}}(A, B)$ is a set.[5]. Then, fix an object $A \in \text{Obj}(\mathcal{C})$. The *covariant Yoneda functor* $h^A : \mathcal{C} \to \mathbf{Set}$ is defined such that for objects $B, C \in \text{Obj}(\mathcal{C})$ and arrows $f : B \to C$,

(1) $h^A(B) \coloneqq \text{Hom}_{\mathcal{C}}(A, B)$,

(2) and $h^A(f) \coloneqq \phi$, where $\phi : \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{C}}(A, C)$ is a map that sends $g \overset{\phi}{\mapsto} f \circ g$.

The notion of identity exists for functors as well. For a category $\mathcal{C}$, the identity functor on $\mathcal{C}$ is the endofunctor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that sends each object and morphism to itself. Now, can you guess when two categories are isomorphic?

**Definition 4.7** (Isomorphism of Categories)**.** Categories $\mathcal{A}$ and $\mathcal{B}$ are said to be isomorphic if there exists functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that

$$F \circ G = 1_{\mathcal{B}} \tag{3}$$

$$G \circ F = 1_{\mathcal{A}} \tag{4}$$

We have notions of injectivity and surjectivity as well.

**Definition 4.8.** A functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *faithful* (resp. *full*) if, for any $A, A' \in \text{Obj}(\mathcal{A})$, $F : \text{Hom}_{\mathcal{A}}(A, A') \to \text{Hom}_{\mathcal{B}}(F(A), F(A'))$ is injective (resp. surjective).

With our new language of functors, we'll prove an analogue of *Cayley's Theorem* from group theory. In case you don't remember it,

**Theorem 4.9** (Cayley's Theorem)**.** *Every finite group is isomorphic to a subgroup of* $S_n$.

[5] Such a category is called *locally small*

Similarly, for categories, we can show that certain small categories can be embedded in concrete categories.

**Theorem 4.10** (Cayley's Theorem for Categories)**.** *Let $\mathcal{C}$ be a category, where the collection of objects and the collection of morphisms are both sets.*[6]*. Then, there exists a category $\overline{\mathcal{C}}$ such that*

*(1) The objects of $\overline{\mathcal{C}}$ are sets,*
*(2) The morphisms of $\overline{\mathcal{C}}$ are mappings,*
*(3) and $\mathcal{C} \cong \overline{\mathcal{C}}$.*

*Proof.* We'll begin by constructing $\overline{\mathcal{C}}$. For objects $C \in \mathcal{C}$, the object $\overline{C} \in \overline{\mathcal{C}}$ is the set

$$\overline{C} = \{f \in \mathcal{C} \mid \exists X \in \operatorname{Obj}(\mathcal{C}) \text{ such that } f \in \operatorname{Hom}_{\mathcal{C}}(X, C)\}^{7}.$$

Notice that for this to be a set, we use that the objects of $\mathcal{C}$ and the arrows $\operatorname{Hom}_{\mathcal{C}}(X, C)$ form sets. Now, for $g \in \operatorname{Hom}_{\mathcal{C}}(C, C')$, the arrow $\overline{g} : \overline{C} \to \overline{C'}$ is a map such that for $f \in \overline{C}$, $\overline{g}(f) := g \circ f$. Naturally, composition will be regular function composition. We leave checking that $\overline{\mathcal{C}}$ does indeed satisfy the remaining parts of Definition 1.1 to the reader. Now, we still have to choose our functors. Given the naming convention we've adopted for the objects and morphisms of $\overline{\mathcal{C}}$, perhaps you can see how we construct $F : \mathcal{C} \to \overline{\mathcal{C}}$ and $G : \overline{\mathcal{C}} \to \mathcal{C}$:

- $F(C) = \overline{C}$ and $F(g) = \overline{g}$,
- $G(\overline{C}) = C$ and $G(\overline{g}) = g$.

Clearly, $F$ and $G$ are inverses. We leave checking that they satisfy the naturality requirements to the reader. ∎

Our new knowledge of functors also allows us to ask about the relationship between a group and its corresponding one object category. Specifically, it turns out

**Proposition 4.11.** *$G$ and $H$ are isomorphic as groups if and only if their associated one object categories are isomorphic.*

*Proof.* We leave this as an exercise for the reader. (Hint: make use of Example 4.5.) ∎

4.2. **Contravariant Functors.** Generally, when we refer to a functor $F$ from categories $\mathcal{A}$ to $\mathcal{B}$, we mean a covariant functor. However, there's another type of functor. Recall Example 1.7.

**Definition 4.12** (Contravariant Functor)**.** A *contravariant functor* from categories $\mathcal{A}$ to $\mathcal{B}$ is a covariant functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. See Figure 4 for a diagrammatic representation.

Let's look at some examples.

**Example 4.13.** Recall Example 4.2. Our functor $F$ : **Set** $\to$ **Set** sent every set to its power set. We'll construct the functor $G$ in a similar way, with only slight modifications to obtain a contravariant version. Specifically, for a set $A$, $A \overset{G}{\mapsto} \mathfrak{P}(A)$ and, for a map $f : A \to B$, $G(f) : \mathfrak{P}(B) \to \mathfrak{P}(A)$ such that $G(f)(X) = f^{-1}(X)$, for $X \subseteq B$.

**Example 4.14.** We'll make a contravariant functor from $\mathbf{Vect}_k$ to $\mathbf{Vect}_k$. For a vector space $V$, send $V$ to its dual space, $V^\vee$. Then, for $f : V_1 \to V_2$, we'll send $f$ to the map $f^\vee : V_2^\vee \to V_1^\vee$ such that, for $g \in V_2^\vee$, $f^\vee(g) = f \circ g$.

[6] Such a category is called *small*

[7] This is technically abuse of notation. We mean to say $f$ is a functor of $\mathcal{C}$

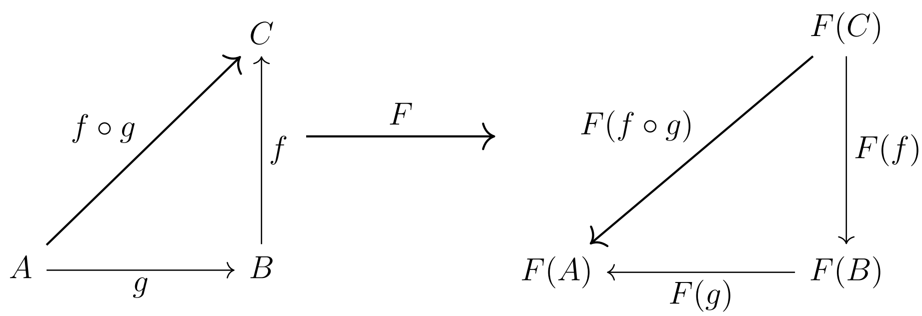


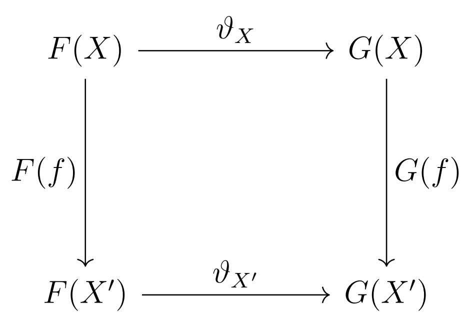

FIGURE 4. $F : \mathcal{A}^{\text{op}} \to \mathcal{B}$ is a functor and $A, B, C \in \text{Obj}(\mathcal{A})$

**Example 4.15** (Contravariant Yoneda Functor)**.** This is the contravariant version of Example 4.6. Once again, consider a category $\mathcal{C}$, where, for all $A, B \in \text{Obj}(\mathcal{C})$, $\text{Hom}_{\mathcal{C}}(A, B)$ is a set. Then, fix an object $A \in \text{Obj}(\mathcal{C})$. The contravariant functor $h_A : \mathcal{C}^{\text{op}} \to \textbf{Set}$ is defined such that for objects $B, C \in \text{Obj}(\mathcal{C})$ and arrows $f : B \to C$,

(1) $h_A(B) \coloneqq \text{Hom}_{\mathcal{C}}(B, A)$,
(2) and $h_A(f) \coloneqq \phi$, where $\phi : \text{Hom}_{\mathcal{C}}(C, A) \to \text{Hom}_{\mathcal{C}}(B, A)$ is a map that sends $g \overset{\phi}{\mapsto} g \circ f$.

4.3. **Natural Transformations.** Now that we've asked what the morphisms between categories are, we can reasonably talk about a category of categories. But, what if also wanted to talk about a category of functors? This is where natural transformations come into play. Think of them as morphisms between functors!

**Definition 4.16** (Natural Transformation)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A *natural transformation* $\vartheta : F \to G$ consists of arrows $\vartheta_X : F(X) \to G(X)$ for $X \in \text{Obj}(\mathcal{A})$ (we call such arrows the components of $\vartheta$ at $X$). Much like functors, we have a *naturality requirement*. Specifically, for $X, X' \in \text{Obj}(\mathcal{A})$ and $f : X \to X'$, we require the diagram



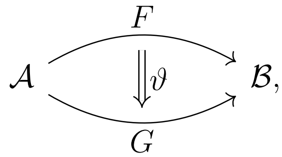

commutes

This makes defining the functor category $[\mathcal{A}, \mathcal{B}]$ easy enough.

(1) Our objects are functors $F : \mathcal{A} \to \mathcal{B}$.
(2) Our morphisms are natural transformations.
(3) The component of the composition is the composition of components.
(4) Our identity morphisms are morphisms where the components are all identity morphisms.

But, what do isomorphisms look like?

**Proposition 4.17.** *For a natural transformation*




*the following are equivalent:*

*(1) $\vartheta$ has a two sided inverse;*
*(2) All components of $\vartheta$ are isomorphisms.*

We call $\vartheta$ satisfying these constraints *natural isomorphisms* and say $F \cong G$.

**Exercise.** Prove this.

## 5. The Yoneda Lemma

Remember the contravariant and covariant Yoneda functors (Examples 4.15 and 4.6)? They will be important here.

**Theorem 5.1** (The Yoneda Lemma)**.** *Let $\mathcal{C}$ be a locally small category*[8] *and $F : \mathcal{C}^{\text{op}} \to \textbf{Set}$ a contravariant functor. Then, there is a map*

$$\text{Hom}_{[\mathcal{C}^{\text{op}},\textbf{Set}]}(h_A, F) \to F(A)$$

*that is a bijection. Moreover,*

$$\text{Hom}(h_\bullet, F) \cong F$$

*as functors.*

What does this mean!? We'll go piece by piece.

**Definition 5.2.** For a locally small category $\mathcal{C}$, the functor $h_\bullet : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \textbf{Set}]$ is defined such that

$$X \overset{h_\bullet}{\mapsto} h_X.$$

Given a morphism $f : A \to B$, $h_\bullet(f) : h_A \to h_B$ is such that

$$h_\bullet(f)_X(x) = f \circ x$$

for $x \in h_A(X)$. We call $h_\bullet$ the *Yoneda embedding*.

**Exercise.** Verify that this is actually a functor.

**Definition 5.3.** For a locally small category $\mathcal{C}$ and functor $F : \mathcal{C}^{\text{op}} \to \textbf{Set}$, $\text{Hom}(h_\bullet, F) : \mathcal{C}^{\text{op}} \to \textbf{Set}$ is a contravariant functor that sends

$$X \overset{\text{Hom}(h_\bullet,F)}{\mapsto} \text{Hom}_{[\mathcal{C}^{\text{op}},\textbf{Set}]}(h_X, F).$$

For morphisms $f : X \to Y$, $\text{Hom}(h_\bullet, F)(f) : \text{Hom}(h_\bullet, F)(Y) \to \text{Hom}(h_\bullet, F)(X)$ is such that

$$(\text{Hom}(h_\bullet, F)(f))(y) = y \circ h_\bullet(f)$$

for $y \in \text{Hom}_{[\mathcal{C}^{\text{op}},\textbf{Set}]}(h_Y, F)$.

**Exercise.** Verify that this is actually a functor.

---

[8]A category is locally small when the collection of morphisms between any two objects is a set

With these two definitions, Theorem 5.1 should at least make sense. Now, instead of seeing a proof, we'll look at two corollaries.

**Corollary 5.4.** *$h_\bullet$ is full and faithful.*

*Proof.* This follows quickly since, by the Yoneda Lemma,

$$|\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathbf{Set}]}(h_A, h_B)| = |h_B(A)| = |\mathrm{Hom}_{\mathcal{C}}(A, B)|$$

■

The faithfulness part of this result is why we call $h_\bullet$ an embedding. Moreover, since it's full, we note that arrows $A \to B$ are "the same as" arrows $h_A \to h_B$. It's also true that our embedding is injective on objects, so we can view $\mathcal{C}$ as a subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

**Corollary 5.5.** *For a locally small category $\mathcal{C}$ and objects $A, B \in \mathrm{Obj}(\mathcal{C})$,*

$$h_A \cong h_B \iff A \cong B$$

*Proof.* This follows as a direct consequence of Corollary 5.4. $h_A \cong h_B \implies A \cong B$ because $h_\bullet$ is full and faithful and $A \cong B \implies h_A \cong h_B$ because $h_\bullet$ is a functor. ■

Informally, Corollary 5.5 tells us that two objects are the same if they're the same from every point of view.

## 6. Further Reading

We recommend [Awo10] for the reader looking for an introduction to category theory. Much of this paper is from material in Awodey's text. [Alu09] may also be an interesting read if the reader would like to see algebra from a category theoretically perspective. Many traditional operations in algebra are actually functors and seeing them as such offers additional insight. [Lei16]'s section on the Yoneda lemma is particularly well done, and, naturally, we recommend it to those interested in a proof and more consequences. Finally, [Rie17] is an extensive approach to the subject. Arguably, it isn't the best introductory text, but it's highly recommended for the experienced reader.

## References

[Alu09] Paolo Aluffi. *Algebra: chapter 0*, volume 104. American Mathematical Soc., 2009.
[Awo10] Steve Awodey. *Category theory*. Oxford university press, 2010.
[DF04] David Steven Dummit and Richard M Foote. *Abstract algebra*, volume 3. Wiley Hoboken, 2004.
[Lei16] Tom Leinster. Basic category theory, 2016.
[Rie17] Emily Riehl. *Category theory in context*. Courier Dover Publications, 2017.

Euler Circle, Palo Alto, CA 94306
*Email address*: akashdhiraj2019@gmail.com